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SOME SEQUENCE SPACES AND ABSOLUTE ALMOST CONVERGENCE

BY

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ABSTRACT. The object of this paper is to introduce a new concept of absolute almost convergence which emerges naturally as an absolute analogue of almost convergence, in the same way as convergence leads to absolute convergence.

1. Introduction. We write throughout x for a sequence $\{x_n\}$ of complex numbers. We write l_∞ and c, respectively, for the Banach spaces of bounded and convergent sequences normed, as usual, by $\|x\| = \sup_{n\geq 0}|x_n|$. We write D for the shift operator; that is

$$D(\{x_n\}) = \{x_{n+1}\}.$$

We recall (see Banach [**1**]) that a Banach limit L is defined as a nonnegative linear functional on l_∞ such that L is invariant under the shift operator (that is, $L(Dx) = L(x)$ for all $x \in l_\infty$) and such that $L(e) = 1$, where $e = (1, 1, \dots)$. Various types of limits, including Banach limits, are considered in Das [**3**]. A sequence $x \in l_\infty$ is said to be almost convergent to the value σ (see Lorentz, [**5**]) if $L(x) = \sigma$ for all Banach limits L; that is, all Banach limits coincide. We denote the set of all almost convergent sequences by $\hat{c}$.

The main object of this paper is to study a new sequence space of absolute almost convergence, which emerges naturally as an absolute analogue of almost convergence, just as absolute convergence emerged out of the concept of convergence. The definition is given in the following section. In §3, we consider spaces $\hat{l}(p)$ which generalise $\hat{l}$ in the same way as $l(p)$ generalises l, space of absolutely convergent sequences. In §4, we consider a related sequence space $\hat{\hat{l}}(p)$ which includes $\hat{l}(p)$.

2. Absolute almost convergence. For any sequence x, write

$$d_{mn} = d_{mn}(x) = \frac{1}{m+1}\sum_{i=0}^{m} x_{n+i}. \tag{2.1}$$

Lorentz [**5**] established the following result.

THEOREM A. *$x \in \hat{c}$ if and only if $d_{mn}(x)$ tends to a limit as $m \to \infty$, uniformly in n.*

It is this characterisation of $\hat{c}$ which enables us to define an absolute analogue of almost convergence.

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Given an infinite series

$$\Sigma a_n, \tag{2.2}$$

which we will denote by a, let

$$x_n = a_0 + a_1 + \cdots + a_n, \tag{2.3}$$

We will suppose throughout that a, x are related by (2.3). (Where no limits are stated, sums throughout are to be taken from 0 to ∞.)

We now extend the definition of $d_{mn}(x)$ to $m = -1$ by taking

$$d_{-1,n} = d_{-1,n}(x) = x_{n-1}. \tag{2.4}$$

We then write, for $m, n \geqslant 0$,

$$\varphi_{mn} = \varphi_{mn}(a) = d_{mn} - d_{m-1,n}. \tag{2.5}$$

A straightforward calculation then shows that

$$\varphi_{on} = a_n; \tag{2.6}$$

$$\varphi_{mn} = \frac{1}{m(m+1)} \sum_{\nu=1}^{m} \nu a_{n+\nu} \qquad (m \geqslant 1). \tag{2.7}$$

We say that the series a (or the sequence x) is absolutely almost convergent[1] if $\Sigma_m|\varphi_{mn}|$ converges uniformly in n. We denote the set of absolutely almost convergent series by $\hat{l}$. We remark that it follows from Theorem A that if x is absolutely almost convergent, then it is almost convergent. The converse of this is false; indeed, even if x is convergent, it need not necessarily be absolutely almost convergent. For it is clear from the definition that absolute almost convergence implies absolute summability $|C, 1|$; but it is well known that there are convergent sequences which are not summable $|C, 1|$.

3. New sequence spaces. We now extend the definition of $\hat{l}$ to a more general space $\hat{l}(p)$ in the same way as l is extended to $l(p)$ (see Simons [7], Bourgin [**2**], Landsberg [**4**] and Maddox [**6**, p. 70]).

Let $p = \{p_m\}$ be a bounded sequence of positive numbers. We write

$$\psi_n = \psi_n(a) = \sum_m |\varphi_{mn}|^{p_m} \tag{3.1}$$

whenever this series converges. We define $\hat{l}(p)$ as the set of series for which (3.1) converges uniformly in n, and $\hat{\hat{l}}(p)$ as the set of series for which (3.1) converges for all n, and ψ_n is bounded. If p_m is a constant (which we will denote also by p), we write $\hat{l}_p$, $\hat{\hat{l}}_p$ in place of $\hat{l}(p)$, $\hat{\hat{l}}(p)$. We omit the suffix p in the case $p = 1$; note that this agrees with the definition of $\hat{l}$ already given.

Now if α_{mn} is any nonnegative real-valued function of two integer variables m and n, there is no relation of implication between the two assertions

(a) $\Sigma_m \alpha_{mn}$ converges uniformly in n;

(b) $\Sigma_m \alpha_{mn}$ converges for all n, and its sum is bounded.

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[1]This concept was introduced by the first author at the British Mathematical Colloquium held at Birmingham in 1968.

But φ_{mn} cannot be chosen arbitrarily; its definition implies certain relations between the values of φ_{mn} for different values of m and n. Thus we cannot exclude the possibility that, when $\alpha_{mn} = |\varphi_{mn}|^{p_m}$, one of (a), (b) might imply the other; and we shall show that, in fact, (a) implies (b). In other words, we have the following theorem.

THEOREM 1. $\hat{l}(p) \subset \hat{\hat{l}}(p)$.

PROOF. Supposing that $a \in \hat{l}(p)$, we have to show that ψ_n is bounded. By definition, there is an integer M such that

$$\sum_{m \geqslant M} |\varphi_{mn}|^{p_m} \leqslant 1. \tag{3.2}$$

Hence it is enough to show that, for fixed m, $|\varphi_{mn}|^{p_m}$ is bounded, or, what is equivalent, that $|\varphi_{mn}|$ is bounded. Now it follows from (3.2) that $|\varphi_{mn}| \leqslant 1$ for $m \geqslant M$ and all n. But, if $m \geqslant 1$,

$$a_{m+n} = (m+1)\varphi_{mn} - (m-1)\varphi_{m-1,n}. \tag{3.3}$$

Applying (3.3) with any fixed $m \geqslant M + 1$, we deduce that a_v is bounded. Hence φ_{mn} is bounded for all m, n, which gives us more than we need. Thus the theorem is proved.

Now write $H = \max(1, \sup p_m)$. Define

$$g_p(a) = \sup_n \{\psi_n(a)\}^{1/H}; \tag{3.4}$$

this exists for $a \in \hat{l}(p)$ in virtue of Theorem 1. It can be proved by "standard" arguments that $g_p(a)$ is a paranorm on $\hat{l}(p)$. Also, with the topology given by this paranorm, the space $\hat{l}(p)$ is complete. If p is a constant sequence, $g_p(a)$ is a norm if $p \geqslant 1$ and a p-norm if $p < 1$ (see [**6**, p. 94]). In these cases, we write $\|a\|_{\hat{p}}$ (or $\|a\|$ where there is no danger of confusion) in place of $g_p(a)$.

THEOREM 2. (i) *If* $p \leqslant \frac{1}{2}$, *then* $a \in \hat{l}_p$ *implies that* $a_n = 0$ $(n \geqslant 1)$.

(ii) *If* $p \geqslant 1$, *then* $l_p \subset \hat{l}_p$, *and this inclusion is proper. Further, if* $a \in l_p$, *then* $\|a\|_{\hat{p}} \leqslant \|a\|_p$, *where* $\|a\|_p$ *is the usual* l_p *norm.*

(iii) *If* $p < 1$, *it is false that* $l_p \subset \hat{l}_p$. (*This is a trivial consequence of* (i) *if* $p \leqslant \frac{1}{2}$.)

(iv) *If* $p > \frac{1}{2}$, *the converse inclusion* $\hat{l}_p \subset l_p$ *is false.*

REMARKS. In the case $p \leqslant \frac{1}{2}$, $a \in \hat{l}_p$ does not imply that $a_0 = 0$. However, (i) shows that in this case $\hat{l}_p$ is trivial. We have not excluded this case because there would have been no gain in simplicity in restricting ourselves to the case $p > \frac{1}{2}$.

We note that the case $p = 1$ of (ii) shows that any absolutely convergent sequence is absolutely almost convergent, but that the converse is false.

PROOF OF THEOREM 2(i). Here we do not need the full force of the hypothesis; we need only the weaker assumption that

$$\sum_m |\varphi_{mn}|^p \tag{3.5}$$

converges for all n. Write

$$(3.6)\qquad T_{mn} = T_{mn}(a) = m(m+1)\varphi_{mn} = \sum_{\rho=n+1}^{m+n} (\rho - n)a_\rho.$$

Thus we are given that, for all $n \geqslant 0$,

$$\sum_{m=1}^{\infty} \frac{|T_{mn}|^p}{m^{2p}} < \infty.$$

Hence

$$(3.7)\qquad \sum_{m=3}^{\infty} \frac{|T_{mn} - 2T_{m-1,n+1} + T_{m-2,n+2}|^p}{m^{2p}}$$
$$\leqslant \sum_{m=3}^{\infty} \frac{|T_{mn}|^p}{m^{2p}} + 2^p \sum_{m=3}^{\infty} \frac{|T_{m-1,n+1}|^p}{m^{2p}} + \sum_{m=3}^{\infty} \frac{|T_{m-2,n+2}|^p}{m^{2p}} < \infty.$$

But

$$T_{mn} - 2T_{m-1,n+1} + T_{m-2,n+2} = a_{n+1},$$

so that, since $p \leqslant \frac{1}{2}$, (3.7) is satisfied only if $a_{n+1} = 0$. Since we may take any $n \geqslant 0$, this gives the conclusion.

PROOF OF THEOREM 2(ii). Suppose that $a \in l_p$. If $m \geqslant 1$ we have, by Hölder's inequality when $p > 1$ and trivially when $p = 1$,

$$|\varphi_{mn}|^p \leqslant \frac{1}{m(m+1)^p} \sum_{\nu=1}^{m} \nu^p |a_{n+\nu}|^p.$$

Hence

$$\sum_{m=1}^{\infty} |\varphi_{mn}|^p \leqslant \sum_{\nu=1}^{\infty} \nu^p |a_{n+\nu}|^p \sum_{m=\nu}^{\infty} \frac{1}{m(m+1)^p} \leqslant \sum_{\nu=1}^{\infty} |a_{n+\nu}|^p.$$

Thus, since $\varphi_{on} = a_n$, we deduce that

$$(3.8)\qquad \sum_m |\varphi_{mn}|^p \leqslant \sum_{\nu=n}^{\infty} |a_\nu|^p.$$

Since uniform convergence of $\sum_m |\varphi_{mn}|^p$ follows at once from (3.8), (ii) of the theorem is now evident, except for the assertion that inclusion is proper; and this is included in (iv).

PROOF OF THEOREM 2(iii). We prove the slightly stronger result that if $p < 1$, we can have $a \in l_p$ without (3.5) converging for any n. For this, let λ be any real constant with $\lambda p > 1$. Let

$$a_n = \begin{cases} r^{-\lambda} & (n = 2^r, n = 1, 2, \ldots); \\ 0 & (\text{otherwise}). \end{cases}$$

Then clearly $a \in l_p$. Taking any fixed n, let $m \geqslant 2n + 2$. There is a ρ which is an integer power of 2 satisfying

$$\tfrac{1}{2}(n + m) < \rho \leqslant n + m.$$

Taking this value of ρ, we deduce from (2.7) that

$$\varphi_{mn} \geqslant \frac{(\rho - n)a_\rho}{m(m+1)}.$$

But

$$\rho - n \geqslant \tfrac{1}{2}(m - n) > \tfrac{1}{4}m,$$

and

$$a_\rho = \left(\frac{\log \rho}{\log 2}\right)^{-\lambda} \geqslant \left(\frac{\log(n+m)}{\log 2}\right)^{-\lambda} \geqslant \left(\frac{\log(3m/2)}{\log 2}\right)^{-\lambda}.$$

Thus, for sufficiently large m, φ_{mn} is greater than or equal to a constant multiple of $1/m(\log m)^\lambda$. Since $p < 1$, the divergence of (3.5) follows.

PROOF OF THEOREM 2(iv). For this we take

$$a_n = \begin{cases} 0 & (n = 0); \\ (-1)^n n^{-1/p} & (n \geqslant 1). \end{cases}$$

Thus $a \notin l_p$. For $m \geqslant 1$,

$$\varphi_{mn} = \frac{1}{m(m+1)} \sum_{\rho=n+1}^{n+m} (-1)^\rho (\rho - n)^{-1/\rho}. \tag{3.9}$$

Now, for $x > 0$,

$$\frac{d}{dx}\left[(x-n)x^{-1/\rho}\right] = \frac{1}{\rho} x^{-1/\rho - 1}\left[(p-1)x + n\right].$$

If $p \geqslant 1$, it follows that $(x - n)x^{-1/\rho}$ is nondecreasing. Hence the sum in (3.9) does not, in modulus, exceed the modulus of its last term, so that

$$|\varphi_{mn}| \leqslant \frac{(m+n)^{-1/\rho}}{m+1} \leqslant \frac{m^{-1/\rho}}{m+1}.$$

Now consider the case in which $\frac{1}{2} < p < 1$. If $n = 0$, then $(x - n)x^{-1/\rho}$ is nonincreasing, so that the sum in (3.9) does not, in modulus, exceed the modulus of the first term. If $n > 0$, then $(x - n)x^{-1/\rho}$ increases for $x < n/(1 - \rho)$ and decreases for $x > n/(1 - \rho)$, the maximum being a constant multiple of $n^{1-1/\rho}$. The sum in (3.9) does not, in modulus, exceed twice this miximum. Since $n^{1-1/\rho} \leqslant 1$ we see that, in any case

$$|\varphi_{mn}| \leqslant \frac{K}{m(m+1)},$$

where K is a constant. The result that $a \in \hat{l}_p$ is now evident.

We now prove a theorem on $\hat{l}(p)$ for general p.

THEOREM 3. *Suppose that for all m, $q_m \leqslant p_m$. Then*

(i) $\hat{l}(q) \subset \hat{l}(p)$;

(ii) *In the topology of $\hat{l}(p)$, $\hat{l}(q)$ is not necessarily a closed subspace of $\hat{l}(p)$. Further, a sequence of elements of $\hat{l}(q)$ which converges to an element of $\hat{l}(q)$ in the topology of $\hat{l}(p)$ need not necessarily do so in the topology of $\hat{l}(q)$.*

PROOF OF (i). Suppose that $a \in \hat{l}(q)$. Then there is an integer M such that, for all n,

$$\sum_{m=M}^{\infty} |\varphi_{mn}|^{q_m} \leqslant 1. \tag{3.10}$$

Hence, for $m \geqslant M$ and all n, $|\varphi_{mn}| \leqslant 1$, so that

$$|\varphi_{mn}|^{p_m} \leqslant |\varphi_{mn}|^{q_m}.$$

The uniform convergence of $\sum|\varphi_{mn}|^{p_m}$ therefore follows from that of $\sum|\varphi_{mn}|^{q_m}$.

PROOF OF (ii). We show that these negative results still hold even when p, q are restricted to be constant sequences. For both clauses, we take $p_m = 2$, $q_m = 1$ (all m), so that $\hat{l}(p)$, $\hat{l}(q)$ are $\hat{l}_2$, $\hat{l}_1$.

For the first clause, let $a^i = \{a^i_n\}$ be defined for $i = 1, 2, 3, \dots$ by

$$a^i_p = \begin{cases} \dfrac{1}{n+1} & (n \leqslant i); \\ 0 & (n > i). \end{cases}$$

Let a be defined by $a_n = 1/(n+1)$. Clearly, $a^i \in \hat{l}_1$ for all i. It is easily proved that $a \in \hat{l}_2$ and that, in the topology of $\hat{l}_2$, $a^i \to a$ as $i \to \infty$. But $a \notin \hat{l}_1$. Thus $\hat{l}_1$ is not closed in $\hat{l}_2$.

For the second clause, define $b^i = \{b^i_n\}$ for $i = 1, 2, 3, \dots$ by

$$b^i_n = \begin{cases} 1 & (n \leqslant i); \\ 0 & (n > i). \end{cases}$$

Then

$$\varphi_{mn}(b^i) = \begin{cases} 1 & (n \leqslant i,\ m = 0); \\ \dfrac{1}{2} & (n < i,\ 1 \leqslant m \leqslant i - n); \\ \dfrac{(i-n)(i-n+1)}{2m(m+1)} & (n < i,\ m > i - n); \\ 0 & (n = i,\ m > 0 \text{ or } n > i, \text{ all } m). \end{cases}$$

We note that, for fixed i, m, $\varphi_{mn}(b^i)$ is a nonnegative nonincreasing function of n; thus (for p or q) sup $\psi_n(a^i)$ is attained for $n = 0$. Hence

$$\|a^i\|_{\hat{1}} = 1 + \tfrac{1}{2}i + \tfrac{1}{2}i(i+1) \sum_{m=i+1}^{\infty} \frac{1}{m(m+1)} = 1 + i;$$

$$\|a^i\|_{\hat{2}} = \left\{1 + \tfrac{1}{4}i + \tfrac{1}{4}i^2(i+1)^2 \sum_{m^2=i+1}^{\infty} \frac{1}{m^2(m+1)^2}\right\}^{1/2} = O(i^{1/2}).$$

Thus, if $\frac{1}{2} < \lambda < 1$, $\{i^{-\lambda}a^i\}$ is a sequence of elements of $\hat{l}_1$ which converges to 0 in the topology of $\hat{l}_2$, but not in the topology of $\hat{l}_1$.

4. Further properties. We now consider some properties of $\hat{\hat{l}}_p$.

THEOREM 4. *Suppose now that p is bounded away from* 0. *Then*

(i) $\hat{\hat{l}}(p)$ *is a complete linear space, paranormed by the function* g_p *defined by* (3.4). *In particular, if* $p \geqslant 1$, $\hat{\hat{l}}_p$ *is a Banach space.*

(ii) $\hat{l}(p)$ *is a closed subspace of* $\hat{\hat{l}}(p)$.

(iii) *If* $p_m \leqslant q_m$ *(all m), then* $\hat{\hat{l}}(p) \subset \hat{\hat{l}}(q)$.

PROOF. (i) may be proved by "standard" arguments, and the details of the proof are therefore omitted. It may, however, be remarked that there is one difference between the proof of (i) and that of the analogous result for $\hat{l}(p)$. As one step in the proof we have to show that, for fixed a, $\lambda a \to 0$ as $\lambda \to 0$ (with the topology given by g_p). If $a \in \hat{l}(p)$ then, given any $\varepsilon > 0$, there is an M such that for all n

$$\sum_{m=M}^{\infty} |\varphi_{mn}(a)|^{p_m} < \varepsilon. \tag{4.1}$$

If $\lambda \leqslant 1$,

$$\sum_{m=M}^{\infty} |\varphi_{mn}(\lambda a)|^{p_m} \leqslant \sum_{m=M}^{\infty} |\varphi_{mn}(a)|^{p_m} < \varepsilon,$$

and since, for fixed M,

$$\sum_{m=0}^{M-1} |\varphi_{mn}(\lambda a)|^{p_m} \to 0$$

as $\lambda \to 0$, this gives the conclusion. If we are given only that $a \in \hat{\hat{l}}_p$, we cannot assert (4.1). We now make use of the assumption that p_m is bounded away from 0 (an assumption which was not made in our investigation of $\hat{l}_p$). There is some constant $\delta > 0$ such that $p_m \geqslant \delta$ (all m). Hence for $|\lambda| \leqslant 1$, $|\lambda|^{p_m} \leqslant |\lambda|^{\delta}$, so that

$$g_p(\lambda a) \leqslant |\lambda|^{\delta} g_p(a).$$

The result clearly follows.

Since, by definition, $\hat{l}(p)$ and $\hat{\hat{l}}(p)$ have the same metric, (ii) follows from the result that $\hat{l}(p)$ is complete.

The proof of part (iii) differs from that of the analogous Theorem 3(i), as we cannot now assert (3.10). If $a \in \hat{\hat{l}}(p)$, then $\sum_m |\varphi_{mn}|^{p_m}$ is bounded. A fortiori, $\varphi_{on} = a_n$ is bounded. It now follows from (2.7) that φ_{mn} is bounded for all m, n; say $|\varphi_{mn}| \leqslant K$. We may suppose that $K \geqslant 1$. Then

$$\sum_m |\varphi_{mn}|^{q_m} \leqslant \sum_m K^{q_m - p_m} |\varphi_{mn}|^{p_m} \leqslant K^H \sum_m |\varphi_{mn}|^{p_m},$$

where $H = \sup_m q_m$. Hence the result.

In Theorem 4, we have imposed the restriction that p is bounded away from 0. To show the need for some such restriction, we now show that if p is unrestricted (apart from the assumptions made thoughout that $p_m > 0$ and the p_m is bounded), then it is not necessarily the case that, with g_p defined by (3.4), $g_p(\lambda a) \to 0$ as $\lambda \to 0$ for fixed $a \in \hat{\hat{l}}(p)$. This is embodied in Theorem 5 below. For this, we require a lemma.

LEMMA. *It is possible to define a with* $a_0 = 0$ *such that*

$$|a_\rho| < \rho^{-1/2} \qquad (\rho \geqslant 1) \tag{4.2}$$

and such that there are increasing sequences of positive integers $\{m_r\}$, $\{n_r\}$ *with*

$$\varphi(m_r, n_r) = 0 \qquad (n_r < m_r); \tag{4.3}$$

$$|\varphi(m_r, n_r)| = A_r \neq 0; \tag{4.4}$$

$$A_r \to 0 \tag{4.5}$$

as $r \to \infty$;

$$|\varphi(m_r, n)| \leqslant A_r^r \qquad (n \geqslant n_{r+1}). \tag{4.6}$$

Here, and in what follows, in order to avoid repeated suffixes, we write $\phi(m, n)$ in place of φ_{mn} whenever m, n are replaced by more complicated expressions. We will also use a similar notation with other letters.

PROOF OF LEMMA. Let $\{\eta_\gamma\}$ be a given sequence of positive numbers decreasing to zero. We define n_1 arbitrarily, and then take

$$n_{r+1} = m_r + n_r + 1; \tag{4.7}$$

this will fix $\{n_r\}$ once $\{m_r\}$ has been chosen. We will choose m_r so that

$$m_{r+1} \geqslant 2m_r + n_r. \tag{4.8}$$

We will choose $a_\rho = 0$ except when, for some r

$$m_r \leqslant \rho \leqslant m_r + n_r. \tag{4.9}$$

It follows from (4.8) that the ranges (4.9) are nonoverlapping. We will arrange that with A_r defined by (4.4),

$$A_r < n_r. \tag{4.10}$$

This will ensure (4.5). We define $T_{mn} = T_{mn}(a)$ by (3.6).

Before giving the definitions of a and of $\{m_r\}$, it is convenient to make some observations. Suppose that, for a given s, $n \geqslant n_{s+1}$. If $r \leqslant s$ then, whatever m, it follows from (4.7) that terms a_ρ with ρ satisfying (4.9) will not occur in the sum defining T_{mn}. Now take $r > s$ and consider, in particular, $m = m_s$. Some terms a_ρ with ρ satisfying (4.9) will occur in the sum defining $T(m_s, n)$ if and only if

$$m_r - m_s \leqslant n < m_r + n_r. \tag{4.11}$$

We note that, for $r > s$,

$$m_r + n_r < m_{r+1} - m_s. \tag{4.12}$$

For, since $\{m_r\}$ is increasing, $m_s < m_r$, so that (4.12) follows from (4.8). Thus the ranges of n given by (4.11) are nonoverlapping. If $n \geqslant n_{s+1}$ and if (4.11) does not hold for any r, then $T(m_s, n) = 0$. If (4.11) holds for some r then

$$T(m_s, n) = \Sigma(\rho - n)a_\rho \tag{4.13}$$

where the sum is taken over

$$\max(n + 1, m_r) \leqslant \rho \leqslant \min(n + m_s, m_r + n_r). \tag{4.14}$$

We now define the sequence $\{m_r\}$ and the values of a_ρ in the range (4.9) inductively. Suppose that m_s has been chosen for $s < r$ (this assumption is omitted in the case $r = 1$); thus n_s is fixed for $s \leqslant r$. Suppose also that a_ρ has been determined for ρ in the range (4.9) with r replaced by s, for all $s < r$. Then for any fixed $n \leqslant n_r$, T_{mn} is a certain constant (which is now fixed) in the range

$$m_{r-1} + n_{r-1} - n \leqslant m < m_r - n;$$

let this constant be denoted by $\tau_r(n)$ (in the case $r = 1$, $\tau_1(n) = 0$ for $n \leqslant n_1$). Now define a_ρ in the range (4.9) so that

$$T(m_r, n) = \begin{cases} 0 & (0 \leqslant n < n_r); \\ 1 & (n = n_r). \end{cases} \tag{4.15}$$

Note that we still have m_r at our disposal; but, once m_r has been chosen, the values of a_ρ for ρ in the range (4.9) are uniquely determined by (4.15). For the case $n = 0$ of (4.15) determine $a(n_r)$; having fixed this, the case $n = 1$ determines $a(n_r + 1)$, and so on.

Now (4.15) ensures that (4.3) and (4.4) are satisfied. Thus it is enough to verify that the remaining conditions will all be satisfied provided that m_r is chosen sufficiently large.

Since $T_{mn} = m(m + 1)\varphi_{mn}$, this is trivial for (4.10). It is also trivial for (4.8) (with r replaced by $r - 1$). Next, for fixed ν with $0 \leqslant \nu \leqslant n_r$, the value of $a(m_r + \nu)$ will depend on the choice of m_r. Regarding $a(m_r + \nu)$ as a function of m_r, it follows easily from (4.15) that, as $m_r \to \infty$,

$$\begin{aligned} a(m_r) &\to \frac{-\tau_r(0)}{m_r}; \\ a(m_r + \nu) &\to \frac{\tau_r(\nu - 1) - \tau_r(\nu)}{m_r} \qquad (0 < \nu < n); \\ a(m_r + n_r) &\to \frac{\tau_r(n_r - 1) - \tau_r(n_r) + 1}{n_r}. \end{aligned} \tag{4.16}$$

Hence (4.2) holds for ρ in the range (4.9) provided that m_r is sufficiently large. Finally, for any $s < r$, consider n satisfying (4.11). Since n_r has been fixed, the number of terms in the sum (4.13) is bounded; thus, again using (4.16), it follows that, uniformly in (4.11),

$$T(m_s, n) = O(1/m_r).$$

Thus, by choosing m_r sufficiently large, we can arrange that, for n in the range (4.11),

$$|\varphi(m_s, n)| \leqslant A_s^s. \tag{4.17}$$

If we choose m_r large enough for (4.17) to hold for all $s < r$, then (4.6) will follow; and the proof of the lemma is thus completed.

We are now in a position to prove

THEOREM 5. *Let $p_n > 0$ and bounded. Then $\hat{l}(p)$ is not necessarily a paranormed space. In fact, there exists $a \in \hat{l}(p)$ such that with g_p defined in* (3.4)

$$g_p(\lambda a) \nrightarrow 0 \quad \text{as } \lambda \to \infty.$$

PROOF. Now let a be defined as in the lemma. We note that (4.2) implies that $|a_p| < 1$, and hence that $|\varphi_{mn}| < 1$ for all m, n. Hence $A_r < 1$. Thus we may define

$$p_m = \begin{cases} \dfrac{1}{\log(1/A_r)} & (m = n_r,\ r = 1, 2, 3, \ldots); \\ 3 & (\text{otherwise}). \end{cases}$$

We will show that, with this definition of p, $a \in \hat{l}(p)$, but that $g(\lambda a) \nrightarrow 0$ as $\lambda \to 0$.

We note that $|\varphi_{on}| = |a_n| < 1$. Also, it follows from (4.2) that, for $m \geqslant 1$,

$$|\varphi_{mn}| \leqslant \frac{1}{m(m+1)} \sum_{\nu=1}^{m} \nu(n+\nu)^{-1/2} \leqslant \frac{1}{m(m+1)} \sum_{\nu=1}^{m} \nu^{1/2} \leqslant Km^{-1/2},$$

where K is a constant. Hence

$$\psi_n(a) \leqslant 1 + K^3 \sum_{m=1}^{\infty} m^{-3/2} + \sum_{r=1}^{\infty} |\varphi(m_r, n)|^{p(m_r)}. \tag{4.18}$$

Thus, in order to prove that $a \in \hat{l}(p)$, it is enough to show that the last sum on the right of (4.18) is bounded. But this sum contains only one term for which neither (4.3) nor (4.6) is applicable. For this term, we have $|\varphi(m_r, n)| < 1$; thus the sum in question is less than

$$1 + \sum_{r=1}^{\infty} A_r^{rp(m_r)} = 1 + \sum_{r=1}^{\infty} e^{-r}$$

by definition of $p(m_r)$. Hence $a \in \hat{l}(p)$. But, if $0 < \lambda < 1$,

$$\psi_{m_r}(\lambda a) \geqslant \varphi_{m_r, n_r}(\lambda a)^{p(m_r)} = |\lambda|^{p(m_r)} A_r^{p(m_r)}$$
$$= e^{-1}\lambda^{p(m_r)} \to e^{-1}$$

as $r \to \infty$, by (4.5) and the definition of $p(m_r)$. Hence

$$(g_p(\lambda a))^H \geqslant 1/e,$$

so that $g_p(\lambda a) \nrightarrow 0$ as $\lambda \to 0$.

This completes the proof.

We now prove the following result.

THEOREM 6. *If, uniformly in n, r,*

$$\sum_{m=0}^{n} \varphi_{mn} = O(1) \tag{4.19}$$

and thus, a fortiori, if $a \in \hat{l}$, then x is bounded.

REMARKS. It is pointed out by Lorentz [5] that an almost convergent sequence is necessarily bounded. It follows, a fortiori, that if $a \in \hat{l}$ then x is bounded. But if we assume only that $a \in \hat{l}$, we do not know that x is almost convergent.

We remark that, if $a \in \hat{l}_p$ with $p > 1$, or even if we make the stronger assumption that $a \in l_p$, then it does not follow that x is bounded. A trivial counterexample is given by $a_n = 1/(n+1)$.

PROOF. A straightforward calculation shows that, for $m \geqslant 1$,

$$x_m - x_0 = \frac{2m+1}{2m} \sum_{\mu=1}^{2m} \varphi_\mu + \sum_{\rho=1}^{m-1} \frac{1}{2m-\rho} \sum_{\mu=1}^{2m-\rho} \varphi_{\mu\rho} - \sum_{\mu=1}^{m} \varphi_{\mu m}.$$

This is clearly bounded whenever (4.19) holds.

Finally, we state without proof a result on matrix transformations. If X, Y are any two sets of sequences, we denote by (X, Y) the set of those matrices $A = (a_{nk})$ which have the property that Aa exists and belongs to Y of every $a \in X$. Write

$$b(n,k,m) = \begin{cases} a_{nk} & (m = 0); \\ \dfrac{1}{m(m+1)} \sum_{\nu=1}^{m} \nu a_{n+\nu,k} & (m \geqslant 1). \end{cases}$$

With this notation, we have the following result.

THEOREM 7. *Let $p \geqslant 1$. Then $A \in (l, \hat{l}_p)$ if and only if $\sum_m |b(n,k,m)|^p$ is bounded for all n, k.*

The proof uses ideas similar to those used (e.g.) in [**6**, p. 167].

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REFERENCES

1. S. Banach, *Théorie des opérations linéaires*, Warszawa, 1932.
2. D. G. Bourgin, *Linear topological spaces*, Amer. J. Math. **65** (1943), 637–659.
3. G. Das, *Banach and other limits*, J. London Math. Soc. (2) **7** (1973), 501–507.
4. M. Landsberg, *Lineare topologische Räume die nicht lokalkonvex sind*, Math. Z. **65** (1956), 104–112.
5. G. G. Lorentz, *A contribution to the theory of divergent sequences*, Acta Math. **80** (1948), 167–190.
6. I. J. Maddox, *Elements of functional analysis*, University Press, Cambridge, 1970.
7. S. Simons, *The sequence spaces $l(p_\nu)$ and $m(p_\nu)$*, Proc. London Math. Soc. (3) **15** (1965), 422–436.

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